

TI in Focus: AP[®] Calculus

2018 AP[®] Calculus Exam: AB-4/BC-4
Scoring Guidelines

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Outline

- (1) Free Response Question
- (2) Scoring Guidelines
- (3) Student performance
- (4) Interpretation
- (5) Common errors
- (6) Specific scoring examples

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H , where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.
- (a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.
- (b) Explain why there must be at least one time t , for $2 < t < 10$, such that $H'(t) = 2$.
- (c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \leq t \leq 10$.
- (d) The height of the tree, in meters, can also be modeled by the function G , given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

$$(a) \quad H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.

$$(b) \quad \frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$$

Because H is differentiable on $3 \leq t \leq 5$, H is continuous on $3 \leq t \leq 5$.

By the Mean Value Theorem, there exists a value c , $3 < c < 5$, such that $H'(c) = 2$.

2 : $\left\{ \begin{array}{l} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{array} \right.$

2 : $\left\{ \begin{array}{l} 1 : \frac{H(5) - H(3)}{5 - 3} \\ 1 : \text{conclusion using} \\ \quad \text{Mean Value Theorem} \end{array} \right.$

- (c) The average height of the tree over the time interval $2 \leq t \leq 10$ is given by $\frac{1}{10-2} \int_2^{10} H(t) dt$.

$$\begin{aligned} \frac{1}{8} \int_2^{10} H(t) dt &\approx \frac{1}{8} \left(\frac{1.5+2}{2} \cdot 1 + \frac{2+6}{2} \cdot 2 + \frac{6+11}{2} \cdot 2 + \frac{11+15}{2} \cdot 3 \right) \\ &= \frac{1}{8} (65.75) = \frac{263}{32} \end{aligned}$$

The average height of the tree over the time interval $2 \leq t \leq 10$ is $\frac{263}{32}$ meters.

- (d) $G(x) = 50 \Rightarrow x = 1$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1+x)100 - 100x \cdot 1}{(1+x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1+x)^2} \cdot \frac{dx}{dt}$$

$$\left. \frac{d}{dt}(G(x)) \right|_{x=1} = \frac{100}{(1+1)^2} \cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\frac{3}{4}$ meter per year.

$$2: \begin{cases} 1: \text{trapezoidal sum} \\ 1: \text{approximation} \end{cases}$$

$$3: \begin{cases} 2: \frac{d}{dt}(G(x)) \\ 1: \text{answer} \end{cases}$$

Note: max 1/3 [1-0] if
no chain rule

Student Performance

Part (a)

- Most students recognized the need for a difference quotient.
- Most used the interval $[5, 7]$. Some used $[3, 7]$.
- Many students were not able to fully interpret $H'(6)$.
- Failed to specify the moment in time, $t = 6$.
- Explanation involved a *rate*.

Part(b)

- Many students knew to use the Mean Value Theorem.
- Did not know to look for a subinterval that led to a secant line slope of 2.
- Including the hypotheses: continuity.
- IVT argument possible.

Student Performance

Part (c)

- Trapezoidal Rule applied to our problem: subintervals of unequal width.
- Missing $\frac{1}{8}$.
- Communication issues: show the work that leads to the answer.

Part (d)

- Good knowledge of the Quotient Rule (or Product Rule).
- Need for the Chain Rule: only $\frac{dG}{dx}$ given.
- Many errors in notation; confusion/interpretation of variables.
- Used $x = 50$ rather than solving $G(x) = 50$.

Part (a) 1: estimate

- (1) Must see a difference and a quotient.
- (2) Bald $\frac{5}{2}$ does not earn the point.

Part (a) 1: interpretation with units

- (1) No eligibility for the second point. 0 - 1 is possible.
- (2) Need three items in the interpretation.
 - Meaning of H connected to the tree (height of tree of just height).
 - Meaning of derivative as a rate of change with units (m/y).
 - Meaning of 6 as $t = 6$ or at 6 years.
- (3) Units could be included with approximation, not explanation.
- (4) Could refer to the tree, rather than the height of the tree.
- (5) Reference to the specific approximation 2.5 OK.
- (6) Alternate phrasing for *rate* and awkward explanation of *time* (no interval).

Part (b) 1: $\frac{H(5) - H(3)}{5 - 3}$

- (1) Need to see a difference and a quotient.
- (2) Response may include words: still need a difference and a quotient.

Part (b) 1: conclusion using Mean Value Theorem

- (1) Eligibility: response must be connected to our problem (interval and secant slope).
- (2) Response must explicitly declare the continuity of H .
- (3) May say H is increasing, point not earned yet.
- (4) Wrong theorem: EVT, IVT; does not earn the second point.
- (5) General formulas: do not earn the second point.
- (6) Alternate method: MVT (twice) and IVT.

Part (c) 1: trapezoidal sum

- (1) This is a method point, not an approximation point.
- (2) Arithmetic and linkage errors come off the approximation point.
- (3) Need to see multiplication leading to correct trapezoid areas and evidence of a sum.
- (4) One error in set-up still earns the first point.

Part (c) 1: approximation

- (1) Earned for our answer only.
- (2) No point for a bald answer.
- (3) No point if linkage errors.

Part (c) Examples

$$(1) \frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3 \quad 1 - ?$$

$$(2) \frac{3.5}{2} + \frac{8}{2} \cdot 2 + \frac{17}{2} \cdot 2 + \frac{26}{2} \cdot 2 \quad 1 - ?$$

$$(3) 1.75 + 8 + 17 + 39 \quad 0 - ?$$

$$(4) \frac{1.5}{2} + \frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3 \quad 0 - 0$$

$$(5) \frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 2 \quad 1 - 0$$

$$(6) \begin{array}{l} \text{left} = 1.5 \cdot 1 + 2 \cdot 2 + 6 \cdot 2 + 11 \cdot 3 = 50.5 \\ \text{right} = 2 \cdot 1 + 6 \cdot 2 + 11 \cdot 2 + 15 \cdot 3 = 81 \end{array} \implies \frac{50.5 + 81}{2} \quad 1 - ?$$

Part (c) Examples

$$(1) \left(\frac{0.5 \cdot 1}{2} + 1 \cdot 1.5 \right) + \left(\frac{2 \cdot 4}{2} + 2 \cdot 2 \right) + \left(\frac{2 \cdot 5}{2} + 2 \cdot 6 \right) + \left(\frac{3 \cdot 4}{2} + 3 \cdot 11 \right)$$

Triangle-rectangle method

1 - ?

$$(2) \frac{1}{8} \left(\frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3 \right)$$

1 - 1

$$(3) \frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3$$
$$= 65.75 = \frac{65.75}{1}$$

1 - 0

Part (d) 2: $\frac{d}{dt}(G(x))$

(1) Correct Product or Quotient Rule *and* correct Chain Rule.

(2) Correct $\frac{dG}{dx}$ with no Chain Rule earns 1 point.

Not eligible for the 3rd point.

Examples

$$\frac{dG}{dx} = \frac{(1+x)100 - 100x}{(1+x)^2} \qquad 1/2 - 0$$

$$\frac{dG}{dx} = 100(1+x)^{-1} + 100x \cdot (-1)(1+x)^{-2} \qquad 1/2 - 0$$

$$\frac{dG}{dx} = \frac{-100x}{(1+x)^2} + \frac{100}{(1+x)} \qquad 1/2 - 0$$

Part (d) 2: $\frac{d}{dt}(G(x))$

Incorrect $\frac{dG}{dx}$ with the Chain Rule *may* earn 1/2 derivative points.

(1) $\frac{dG}{dx}$ must *look like* an attempt at the Quotient or Product Rule.

(2) $\frac{dG}{dx}$ must be a rational function in x .

It cannot simplify to a constant or polynomial expression.

(3) A single error in $\frac{dG}{dx}$ with the Chain Rule may be eligible for the 3rd point.

Read for a consistent answer in only four cases.

(4) Other errors in $\frac{dG}{dx}$ with the Chain Rule:

May earn 1/2 but not eligible for the third point.

(5) Notation errors: If a correct Quotient or Product Rule, 1 point.

Third point for our answer only.

Part (d) 1: answer

(1) Must be our answer, except in four special cases.

(2) A single error in $\frac{dG}{dx}$, with the Chain Rule.

Examples

$$(1) \frac{dG}{dt} = \frac{(1+x)100 + 100x}{(1+x)^2} \cdot \frac{dx}{dt} \implies \frac{dG}{dt} = \frac{9}{4} \quad 1/2 - 1$$

$$(2) \frac{dG}{dt} = \frac{(1+x)100 - 100x}{1+x^2} \cdot \frac{dx}{dt} \implies \frac{dG}{dt} = \frac{3}{2} \quad 1/2 - 1$$

$$(3) [100(1+x)^{-1} + 100x(1+x)^{-2}] \cdot \frac{dx}{dt} \implies \frac{dG}{dt} = \frac{9}{4} \quad 1/2 - 1$$

$$(4) [100(1+x)^{-1} - 100x \cdot -1(1+x)^{-2}] \cdot \frac{dx}{dt} \implies \frac{dG}{dt} = \frac{9}{4} \quad 1/2 - 1$$

Part (d) Examples

Reasonable attempts at the Quotient Rule

$$(1) \frac{dG}{dt} = \frac{(1+x)100 - 100x}{1+x} \cdot \frac{dx}{dt} \quad 1/2 - 0$$

$$(2) \frac{dG}{dt} = \frac{(1+x)(100) \frac{dx}{dt} - (100x) x \frac{dx}{dt}}{(1+x)^2} \quad 1/2 - 0$$

Notation issues

$$(1) \frac{dG}{dt} = \frac{(1+x) \left(100 \frac{d}{dx} \right) - \left(\frac{d}{dx} \right) (100x)}{(1+x)^2} \quad 1/2 - ?$$

$$(2) \frac{dg}{dh} = \frac{100(1+x) - (100x)}{(1+x)^2} dx \quad 1/2 - ?$$

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