

TI in Focus: AP[®] Calculus

2020 Mock AP[®] Calculus Exam

BC-2: Solutions, Concepts, and Scoring Guidelines

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BC 2

t	0	2	6	8	10	12
$y'(t)$	4	8	-2	3	-1	-5

A particle moves in the coordinate plane with position $(x(t), y(t))$ at time t , where t is measured in seconds and $x(t)$ and $y(t)$ are twice-differentiable functions, both measured in meters.

For all times t , the x -coordinate of the particle's position has derivative

$$x'(t) = \frac{t}{\sqrt{t^2 + 25}}.$$

Selected values of $y'(t)$, the derivative of $y(t)$, over the interval $0 \leq t \leq 12$ seconds are shown in the table.

The position of the particle at time $t = 12$ is $(x(12), y(12)) = (4, -3)$.

- (g) Given $y''(12) = -2$ and $y'''(12) = 8$, find the third-degree Taylor polynomial approximation for y about $t = 12$.

Key Concepts

Power Series

If the function f has a power series expansion at a , then it must be of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \\ &= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \dots \end{aligned}$$

This series is called the **Taylor series of the function f at a** (or **about a** or **centered at a**).

For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

This special case is called a **Maclaurin series**.

Solution

$$y(12) = -3$$

y-coordinate of the position at time $t = 12$

$$y'(12) = -5$$

Value of $y'(t)$ in the table

$$y''(12) = -2 \quad \text{and} \quad y'''(12) = 8$$

Given in part (g)

$$g(t) = y(12) + y'(12)(t - 12) + \frac{y''(12)}{2!}(t - 12)^2 + \frac{y'''(12)}{3!}(t - 12)^3$$

$$= -3 + (-5)(t - 12) + \frac{-2}{2}(t - 12)^2 + \frac{8}{6}(t - 12)^3$$

$$= -3 - 5(t - 12) - (t - 12)^2 + \frac{4}{3}(t - 12)^3$$

Scoring Guidelines

(g) Let $g(t)$ be the third degree Taylor polynomial for y at $t = 12$.

$$\begin{aligned}g(t) &= y(12) + y'(12)(t - 12) + \frac{y''(12)}{2!}(t - 12)^2 + \frac{y'''(12)}{3!}(t - 12)^3 \\&= -3 + (-5)(t - 12) + \frac{-2}{2}(t - 12)^2 + \frac{8}{6}(t - 12)^3 \\&= -3 - 5(t - 12) - (t - 12)^2 + \frac{4}{3}(t - 12)^3\end{aligned}$$

2: $\left\{ \begin{array}{l} 1 : \text{two terms of the Taylor} \\ \text{polynomial} \\ 1 : \text{remaining terms} \end{array} \right.$

Scoring Notes

1: two terms of the Taylor polynomial

- Earned for any two correct terms, signs included.
- Only polynomials centered at $t = 12$ (or $x = 12$) are eligible for any points.

1: remaining terms

- Earned for the correct remaining two terms.
- Including terms of degree greater than 3 or adding $+\dots$ to the Taylor polynomial presented does not earn the second point.
- Simplification error: does not earn the second point.
- Can name this polynomial almost anything, but cannot equate it to $y(t)$.
- Cannot equate the polynomial to a constant, for example, $y(12)$ or $T_3(12)$.

- (h) Suppose that over the time interval $[12, 15]$ the y -coordinate of the position of the particle is the same as the Taylor polynomial approximation found in part (g). Set up but do not evaluate an expression that represents the total distance traveled by the particle over the interval $[12, 15]$.

Key Concepts

Distance Traveled by a Particle

Suppose a particle moves in the plane so that its position at time t is given by the parametric equations $x = f(t)$ and $y = g(t)$.

Consider the vector function $\mathbf{r} = \langle f(t), g(t) \rangle$.

$\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t))$.

If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$ then $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$

The **velocity vector** $\mathbf{v}(t)$ is given by

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

The total distance traveled by the particle from time $t = a$ to $t = b$ is

$$\text{distance traveled} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Solution

For $12 \leq t \leq 15$:

$$y(t) = g(t) = -3 + (-5)(t - 12) - (t - 12)^2 + \frac{4}{3}(t - 12)^3 \quad \text{Equality given; } g(t) \text{ from part (g)}$$

$$x'(t) = \frac{t}{\sqrt{t^2 + 25}} \quad x'(t) \text{ given in statement of the problem}$$

$$y'(t) = g'(t) = -5 - 2(t - 12) + 4(t - 12)^2 \quad \text{Derivative term-by-term}$$

The distance traveled by the particle over the time interval $[12, 15]$ is

$$\int_{12}^{15} \sqrt{\left[\frac{t}{\sqrt{t^2 + 25}}\right]^2 + [-5 - 2(t - 12) + 4(t - 12)^2]^2} dt$$

Scoring Guidelines

(h) For $12 \leq t \leq 15$,

$$y(t) = g(t) = -3 - 5(t - 12) - (t - 12)^2 + \frac{4}{3}(t - 12)^3$$

$$x'(t) = \frac{t}{\sqrt{t^2 + 25}}$$

$$y'(t) = g'(t) = -5 - 2(t - 12) + 4(t - 12)^2$$

The distance traveled by the particle over the time interval $[12, 15]$ is

$$\int_{12}^{15} \sqrt{\left[\frac{t}{\sqrt{t^2 + 25}}\right]^2 + [-5 - 2(t - 12) + 4(t - 12)^2]^2} dt$$

2: $\begin{cases} 1 : \text{expression for } y'(t) \\ 1 : \text{expression for distance traveled} \end{cases}$

Scoring Notes

1: expression for $y'(t)$

- Earned for the correct derivative $y'(t)$.
- Can import an incorrect third-degree Taylor polynomial about $t = 12$.
- If $y'(t)$ is incorrect, still eligible for the second point.

1: expression for distance traveled

- Earned for a correct definite integral.
- Simplification error: second point is not earned.
- dx versus dt , and missing dt

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