Math Objectives

- Students will be able to numerically approximate solutions to logarithmic equations.
- Students will be able graphically determine exact solutions to logarithmic equations using the functions $f(x) = \log_a(x)$ and $f^{-1}(x) = a^x$ and the composition $f \circ f^{-1}(x) = x$.
- Students will be able find exact solutions to exponential equations using algebraic techniques that employ the relationship $f^{-1} \circ f(x) = x$.
- Students will reason abstractly and quantitatively (CCSS Mathematical Practice).

Vocabulary

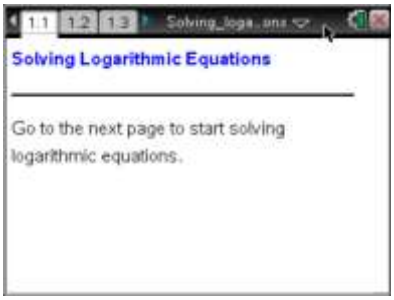
- exponential functions and equations
- logarithmic functions and equations
- inverse functions
- composition of functions

About the Lesson

- This lesson involves numeric, graphical, and algebraic solutions to the equation $\log_3 x = 1.5$.
- Students will:
  - Analyze numeric patterns, predict an approximate solution, and evaluate predictions in a spreadsheet.
  - Consider the graphs of both $f(x) = \log_3 x$ and $f^{-1}(x) = 3^x$ to determine that $f(x) = 1.5$ precisely when $f^{-1}(1.5) = x$.
  - Use the compositional relationship of $\log_3 3^x = x$ to solve the equation. Since $\log_3 3^{1.5} = 1.5$, the solution to the equation $\log_3 x = 1.5$ is $x = 3^{1.5}$.
  - Consider composition in the opposite order, solving the equation algebraically by employing $3^{\log_3 x} = x$
  - Use these techniques to solve similar equations.

TI-Nspire™ Navigator™ System

- Use Screen Capture to share numeric spreadsheet data.
- Use Quick Poll to discuss student responses.
- Use Live Presenter to analyze graphs.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Enter a spreadsheet value
- Operate a minimized slider
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- You can hide the entry line by pressing ctrl G.

Lesson Materials:

Student Activity

Solving_Logarithmic_Equations_Student.pdf

Solving_Logarithmic_Equations_Student.doc

TI-Nspire document

Solving_Logarithmic_Equations.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

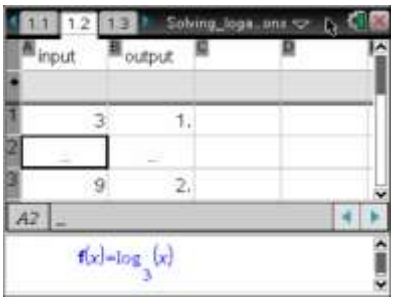
1. Estimate the solution to the equation $\log_3 x = 1.5$ using the following numeric pattern.

   $\log_3 3 = 1$
   $\log_3 x = 1.5$
   $\log_3 9 = 2$

   Sample answer: $x \approx 6$

TI-Nspire Navigator Opportunity: *Quick Poll (Open Response)*
See Note 1 at the end of this lesson.

Move to page 1.2.



2. The table shows inputs and outputs for the function $f(x) = \log_3 x$.

   a. Input your estimate from question 1 into cell A2. Input other values to get the output as close as possible to 1.5. Record your closest input and output below.

   Sample answer:

   $f(3) = 1$
   $f(5.196) = 1.49997$
   $f(9) = 2$

   b. Is there an input value that results in an output value of exactly 1.5?

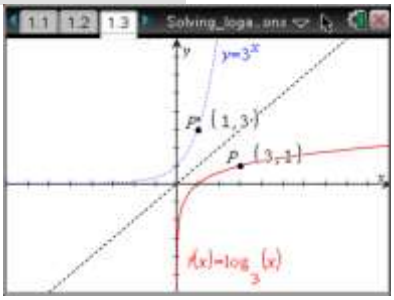
   Answer: No, there is not an exact value.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 2 at the end of this lesson.

Teacher Tip: Students should try to enter more and more decimal places but they will never be able to get exactly 1.5 (unless they enter $3^{1.5}$). If an approximate solution of 5.19613 is put into the spreadsheet, the exact output appears to be 1.5. As a class, if necessary, briefly discuss how the handheld is rounding.

Tech Tip: If students experience difficulty dragging a point, check to make sure that they have moved the cursor until it becomes a hand (☜) getting ready to grab the point. Then press ctrl ☜ to grab the point and close the hand (✋).

Move to page 1.3.



3. The graph of the function $f(x) = \log_3 x$ is shown along with its inverse $f^{-1}(x) = 3^x$. Point P' is the reflection of point P over the line $y = x$.

   a. Suppose the coordinates of P are $(3, 1)$. Write an exponential equation by substituting the coordinates of P' into the function $f^{-1}(x) = 3^x$.

   Answer: $3^1 = 3$

   b. Move point P' so that the input $f^{-1}(x) = 3^x$ is 1.5. According to the graph, what is the *approximate* solution to the equation $\log_3 x = 1.5$? Why is this an approximate solution?

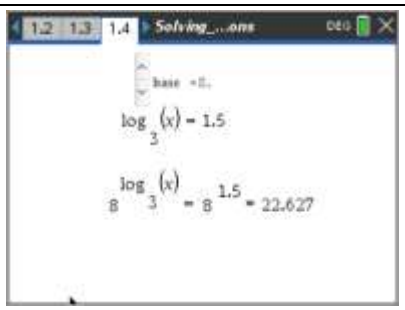
   Answer: When the input of $f^{-1}(x)$ is 1.5, the output is $3^x \approx 5.19615$. This means that the approximate solution is 5.19165. This solution is approximate only because the output is being given in decimal form and is rounded. The exact answer is $3^{1.5}$.

   c. Recall that the composition of any function and its inverse always results in *x*. In other words, $f \circ f^{-1}(x) = f\left(f^{-1}(x)\right) = x$. As such, the composition of $f(x) = \log_3 x$ and $f^{-1}(x) = 3^x$ results in the equation $\log_3 3^x = x$. Use this composition relationship to find the *exact* solution to the equation $\log_3 x = 1.5$. What is the exact solution?

   Answer: $3^{1.5}$

TI-Nspire Navigator Opportunity: *Live Presenter*
See Note 3 at the end of this lesson.

Move to page 1.4.



4. Solve the equation $\log_3 x = 1.5$ by changing the base and reducing the left side of the equation to *x*. To change the base, click the up and down arrows.
   a. What base results in the exact solution?

      Answer: Base 3

   b. What is the exact solution to $\log_3 x = 1.5$? Why?

      Answer: The exact solution is $x = 3^{1.5}$. For $f(x) = \log_3(x)$, $f^{-1}(x) = 3^x$, and $f\left(f^{-1}(x)\right) = x$. Thus, $\log_3 3^{1.5} = 1.5$.

> **Teacher Tip:** Students might need some help with this page to understand exactly what is being displayed. You may want to go through an example first. Make sure to point out that the logarithmic expression is the exponent.

TI-Nspire Navigator Opportunity: *Live Presenter*
See Note 4 at the end of the lesson.

Move to page 1.5.

5. You found an approximate numeric solution in questions 1 and 2, an exact graphical solution in question 3, and an exact numerical solution in question 4. Compare your solutions using the Calculator page provided. How do your solutions compare?

   Answer: By calculating $3^{1.5}$ directly in a Calculator page, students should get an approximate answer of 5.19615, which matches the other solutions.

TI-Nspire Navigator Opportunity: *Live Presenter*
See Note 5 at the end of the lesson.

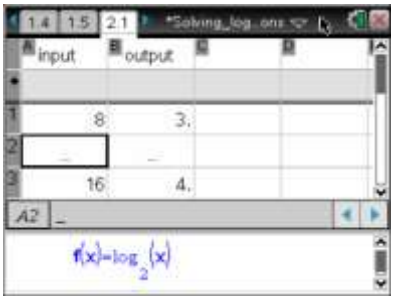
6. Estimate the solution to the equation $\log_2 x = 3.2$ using the following numeric pattern.

$\log_2 8 = 3$
$\log_2 x = 3.2$
$\log_2 16 = 4$

Sample answer: $x \approx 10$

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 6 at the end of the lesson.

Move to page 2.1.

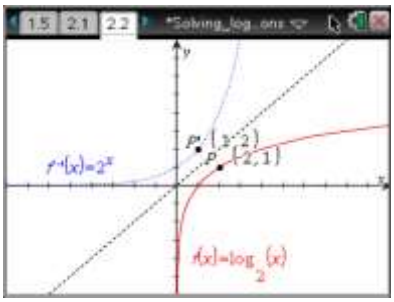


7. The table shows inputs and outputs for the function $f(x) = \log_2 x$.

Input your estimate from question 6 into cell A2. Input other values to get the output as close as possible to 3.2. Record your closest input and output below:

Sample answer:
$f(8) = 3$
$f(\underline{9.189}) = \underline{3.19991}$
$f(16) = 4$

Move to page 2.2.



8. The graph of $f(x) = \log_2 x$ is shown along with its inverse $f^{-1}(x) = 2^x$. Point P' is the reflection of point P over the line $y = x$. Move point P' so that the input of $f^{-1}(x) = 2^x$ is 3.2.

According to the graph, what is the *approximate* solution to the equation $\log_2 x = 3.2$?

Answer: When the input of f^{-1} is 3.2, the output is $2^{3.2} \approx 9.18959$.
This means that the approximate solution is 9.18959.

Move to page 2.3.

9. Solve the equation $\log_2 x = 3.2$ by changing the base and reducing the left side of the equation to *x*. To change the base, click the up and down arrows.

   a. What base results in the exact solution?

      Answer: Base 2.

   b. What is the exact solution to $\log_2 x = 3.2$? Why?

      Answer: The exact solution is $x = 2^{3.2}$. For $f(x) = \log_2(x)$, $f^{-1}(x) = 2^x$, and $f\left(f^{-1}(x)\right) = x$. Thus, $\log_2 2^{3.2} = 3.2$.

TI-Nspire Navigator Opportunity: *Quick Poll (Open Response)*
See Note 7 at the end of the lesson.

Move to page 2.4.

10. Determine how close your estimates from questions 7 and 8 were by entering your exact answer from question 9 in this Calculator page. How do your solutions compare?

    Answer: By calculating $2^{3.2}$ directly in a Calculator page, students should get an approximate answer of 9.18959, which matches the other solutions.

11. Use the algebraic method from questions 4 and 9 to find exact solutions to these equations.

    a. $\log_5 x = 1.3$

    Answer: $5^{1.3}$

    b. $\log_7 x = \sqrt{2}$

    Answer: $7^{\sqrt{2}}$

c. $\log_6 x = -\frac{10}{9}$

Answer: $6^{-10/9}$

Teacher Tip: Pages 3.1, 4.1, and 5.1 are set up with sliders if you choose to let the students use these pages.

Teacher Tip: The last problems are provided as assessment. If you do not want students to use these last pages (3.1, 4.1, 5.1), delete them before giving the file to students.

Tech Tip: To remove the last three pages, press ctrl ▲. Arrow to the pages to delete and press del on each page to clear. Resave with the same or another name.

Extension: Students have solved equations of the form $\log_b x = a$, where $a,b \in \mathbb{R}, b > 0, b \neq 0$. Ask students why the conditions on *a* and *b* are necessary.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- Logarithmic equations can have exact or approximate decimal or non-decimal solutions.
- Non-decimal solutions can be approximated graphically using the inverse relationship between exponential and logarithmic functions.
- Exact solutions can be found graphically using the inverse relationship between exponential and logarithmic functions.
- Exact solutions can also be found algebraically using the inverse relationship between logarithmic and exponential functions and the compositional relationship between inverse functions.

Assessment

Question 11 could be used as an assessment item. Students could also be asked to solve multiple-step equations such as $\log_5 x + 6 = 8$ or $\log_7 (x + 2) = 9$.

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Note 1

Question 1, *Quick Poll (Open Response):* Send an Open Response Quick Poll asking students to submit their answers to question 1.

Note 2

Question 2, *Screen Capture*: Take a Screen Capture to allow students to share their inputs and outputs.

Note 3

Question 3, *Live Presenter:* Live Presenter can be used to show students how to grab and drag the point *P*' as well as to make sure students remember the relationship between the ordered pairs of two functions that are inverses of each other.

Note 4

Question 4, *Live Presenter:* Live Presenter can be used to show students how to operate a minimized slider to change the base of the exponent.

Note 5

Question 5 *Live Presenter:* Live Presenter can be used to show students how to enter the base and the argument for the exponential expression. .

Note 6

Question 6 *Quick Poll (Open Response):* Send an Open Response Quick Poll asking students to submit their answers to question 6.

Note 7

Question 9 *Quick Poll (Open Response):* Send an Open Response Quick Poll asking students to submit their answers to question 9.