# Pascal's Triangle

TI-84PlusCE™   Investigation   Student   50 min

## Introduction

French mathematician Blaise Pascal [1623 – 1662] is not the creator of this famous triangle, the history of the triangle dates back through Chinese mathematics where it is still referred to as Yang Hui's triangle, and even further back to the Greeks [BC]. The title "Pascal's Triangle" was accredited posthumously by another French Mathematician Pierre Raymond de Montmort.

On the surface Pascal's triangle generates a set of numbers useful to probability and binomial expansion; however a whole treasure chest of patterns are hidden in this amazing triangle.

## Generating the Triangle

Watch the PowerPoint presentation on Pascal's triangle.

Watch to see how each successive number is produced on slide 1.

### Question: 1

The first row in Pascal's triangle is Row zero (0) and contains a one (1) only. The PowerPoint animation reveals rows 0 through to 4. Draw these rows and the next three rows in Pascal's triangle.
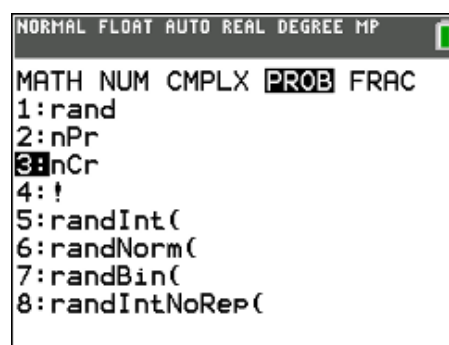
## Combinatorics and Polynomial Expansions

The probability menu contains the combinatorics command $^{n}C_{r}$

[ Math ] > Prob > nCr

Calculate each of the following:

$$^{5}C_{0} , \ ^{5}C_{1} , \ ^{5}C_{2} \ ... \ ^{5}C_{5}$$

### Question: 2

How do the calculations $^{5}C_{0} , \ ^{5}C_{1} , \ ^{5}C_{2} \ ... \ ^{5}C_{5}$ relate to the fifth row of Pascal's triangle?
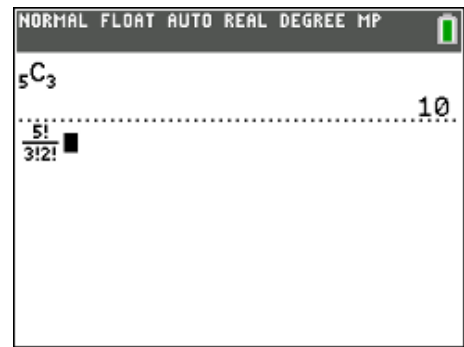
Author: P. Fox

Compare the expression:

$$\frac{n!}{r!(n-r)!}$$

With those from the $^nC_r$ calculations.

**Example**: $^5C_3 = 10$  as shown opposite has n = 5 and r = 3.

The (!) sign is called 'factorial' and can found in the probability menu immediately below the combinatorics command.

```
NORMAL FLOAT AUTO REAL DEGREE MP
₅C₃
                                    10
  5!
 ───
 3!2! ■
```

## Question: 3

Use the formula:   $\dfrac{n!}{r!(n-r)!}$  to calculate the 7th row of Pascal's triangle.

**Tip**

To generate an entire row of Pascal's triangle, store the numbers in a list: { 0,1, 2, 3, 4, 5, 6, 7 } → L₁. then use $^7C_{L_1}$ to see the entire row generated at once. This also works with the formula.

The sequence command can also be used: nCr(8,seq(x,x,0,8)) to generate the 8th row.

## Question: 4

Write down the coefficients in each of the expansions below and locate these sets of numbers in Pascal's triangle?

a)  $(x+y)^2$

b)  $(x+y)^3$

c)  $(x+y)^4$

## Question: 5

Use Pascal's triangle to identify the coefficients:  $a_0, a_1 ... a_6$  in the expansion of  $(x+y)^6$  as shown below:

$$(x+y)^6 = a_0 x^6 + a_1 x^5 y + a_2 x^4 y^2 + a_3 x^3 y^3 + a_4 x^2 y^4 + a_5 xy^5 + a_6 y^6$$

# Triangular Numbers

Triangular numbers are essentially any number that can be represented by a dot pattern forming a triangle. Consider the triangle opposite; it contains 10 marbles (dots), so 10 is a triangular number. Remove the bottom row and you will have 6 marbles, so 6 is a triangular number. Add another row to the bottom of this triangle and it will contain 15 marbles, so 15 is a triangular number. The first four triangular numbers are: 1, 3, 6 and 10.
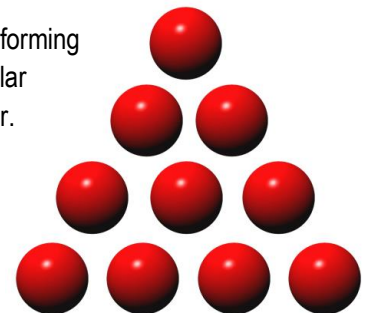
## Question: 6

Write down the first 10 triangular numbers.

Author: P. Fox

**TEXAS INSTRUMENTS**

Watch the PowerPoint presentation on Pascal's triangle – triangular numbers to see how successive triangular numbers are created followed by a series of slides that provides some visual assistance to help establish a formula.

n = 4

Triangular Numbers

**Question: 7**

After viewing the PowerPoint slides, including the 'establishing a formula slides', write a formula for Triangular numbers.

**Question: 8**

The triangular numbers are in Pascal's triangle (shown opposite).

a)   In which row is the fourth Triangular number?

b)   Use combinatorics to write an expression for the $n^{th}$ Triangular number and compare this formula to the one created in Question 7.

```
        1
      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1
1   5  10  10   5   1
1   6  15  20  15   6   1
1   7  21  35  35  21   7   1
```

**Question: 9**

Triangular numbers are the same as the sum of the first n natural numbers. The calculator can compute the sum of a set of numbers easily.

a)   Use sigma notation ($\Sigma$) to calculate the sum of the first 3 numbers. $(1 + 2 + 3) = \sum_{n=1}^{3} n$

**Hint**:  Use [Alpha] [F2] to access the sigma notation.

b)   Use the sigma notation to calculate the sum of the first 8 numbers.

c)   Use the formula generated from the previous questions to check the answers produced here.

# Tetrahedral Numbers

The Tetrahedral numbers are a three dimensional version of the triangular numbers. A tetrahedron is a triangular pyramid. In the pyramid shown opposite the top row contains one ball. The second row contains three providing a total of four balls. The third row contains six balls for a total of ten.

The first three tetrahedral numbers are therefore: { 1, 4, 10 }
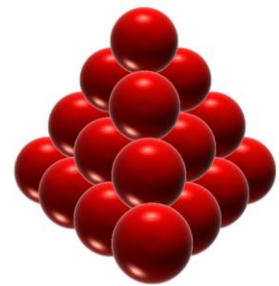
**Question: 10**

Tetrahedral numbers are a figurate number representing the total number of objects required to make a tetrahedron.

a)   The first four tetrahedral numbers are: { 1, 4, 10, 20 }. Where do the tetrahedral numbers appear in Pascal's triangle?

b)   The tetrahedral numbers can be computed as the sum of the triangular numbers. The fifth tetrahedral number can be computed as: $(1 + 3 + 6 + 10 + 15) = \sum_{n=1}^{5} \dfrac{n(n+1)}{2}$ .

Use the sigma notation to determine the $6^{th}$ tetrahedral number.

d) Use sigma notation (Σ) to help determine a formula for the tetrahedral numbers.
   **Hint**: Use the formula computed for triangular numbers in the sum and plot them on a graph.

e) Given the location of the tetrahedral numbers in Pascal's triangle, determine the formula for the tetrahedral numbers using combinatorics.

## Powerful Numbers

Pascal's triangle is created by adding pairs of numbers to create elements in the next row, but what happens if you add all the numbers in each row? Rows zero through to four are shown opposite.

```
        1              1
      1 + 1            2
    1 + 2 + 1          4
  1 + 3 + 3 + 1        8
1 + 4 + 6 + 4 + 1     16
```
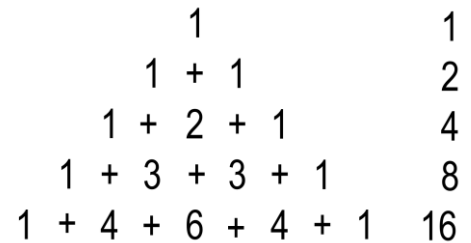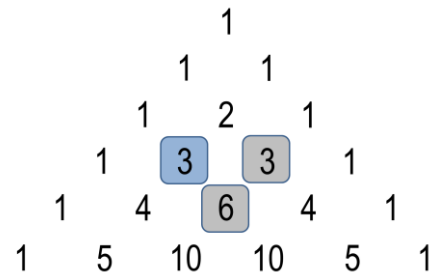
### Question: 11

Generate the values in the 10th row of Pascal's triangle, calculate the sum and confirm that it fits the pattern.

### Question: 12

Given the relationship between the coefficients of $(x+y)^n$ and Pascal's triangle, explain why the sum of each row produces this set of numbers.

## Square Numbers

Each row of Pascal's triangle is generated by repeated and systematic addition. The result of this repeated addition leads to many multiplicative patterns. Another relationship in this amazing triangle exists between the second diagonal (natural numbers) and third diagonal (triangular numbers). If you square the number in the 'natural numbers' diagonal it is equal to the sum of the two adjacent triangular numbers (shown opposite).

$3^2 = 6 + 3$

The same is true for the next number in the natural numbers diagonal:

$4^2 = 10 + 6$

### Question: 13

Show that this relationship holds for the next 3 numbers in the natural numbers diagonal.
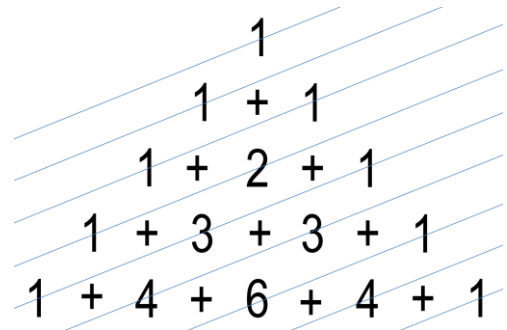
### Question: 14

Show this relationship holds for all numbers in the 'natural numbers diagonal' and the 'triangular numbers' diagonal.

Author: P. Fox

## Shallow Diagonals

Shallow diagonals can cross multiple rows in Pascal's triangle. The triangle must be drawn accurately (as shown) to identify all the numbers that lie on the shallow diagonals.

| Diagonal: | Numbers: | Sum: |
|:---:|:---:|:---:|
| 1 | 1 | 1 |
| 2 | 1 | 1 |
| 3 | 1, 2 | 3 |
| 4 | 1, 3, 1 | 5 |
| 5 | 1, 4, 3 | 8 |



### Question: 15

Determine the numbers that lie on the next two diagonals, their corresponding sum and the pattern that is being formed with the sums of these diagonals.

## Exhaustive Products

Some patterns in Pascal's triangle are easier to find and prove than others. This pattern is one of the most amazing hidden gems in Pascal's triangle. The rule for this pattern is to find the product of the numbers in row n, and multiply this by the product of the numbers in row n + 2, then, divide the result by the product squared for the numbers in row n + 1. This rule can be written as:

$$\frac{\prod_{r=0}^{n-1}\left(^{n-1}C_r\right) \times \prod_{r=0}^{n+1}\left(^{n+1}C_r\right)}{\left(\prod_{r=0}^{n}\left(^{n}C_r\right)\right)^2}$$

For example, suppose $n = 5$.

Row 4 = { 1, 4, 6, 4, 1 }   $\prod_{r=0}^{n-1}\left(^{n-1}C_r\right) = \prod_{r=0}^{4}\left(^{4}C_r\right) = 1 \times 4 \times 6 \times 4 \times 1 = 96$

Row 5 = { 1, 5, 10, 10, 5, 1 }   $\prod_{r=0}^{n}\left(^{n}C_r\right) = \prod_{r=0}^{5}\left(^{5}C_r\right) = 1 \times 5 \times 10 \times 10 \times 5 \times 1 = 25,000$

Row 6 = { 1, 6, 15, 20, 15, 6, 1 }   $\prod_{r=0}^{n+1}\left(^{n+1}C_r\right) = \prod_{r=0}^{6}\left(^{6}C_r\right) = 1 \times 6 \times 15 \times 20 \times 15 \times 6 \times 1 = 162,000$

$$\frac{96 \times 162,000}{25,000^2} \approx \frac{7776}{3125} \approx 2.48832$$

**Tip**

The product command is in the List menu and also the catalogue.

The syntax for the product command is:  prod( list, [start], [end]).  Start and End are optional.

Author: P. Fox

TEXAS INSTRUMENTS

**Question: 16**

Determine the value of the products and resulting approximation when $n$ = 10.

**Question: 17**

The values in this product get very big, very quickly. The maximum value the calculator can compute is for the 67th, 68th and 69th rows. It can be shown however that:

$$\frac{\prod_{r=0}^{n-1}\left({}^{n-1}C_r\right)\times\prod_{r=0}^{n+1}\left({}^{n+1}C_r\right)}{\left(\prod_{r=0}^{n}\left({}^{n}C_r\right)\right)^2}=\left(1+\frac{1}{n}\right)^n$$

a)   Compare the result from this algebraic equivalence by substituting n = 10.

b)   It is possible to substitute larger values into this algebraic equivalence. What is the ratio when n = 1000.
     **Note**: This is equivalent to finding the product of the terms in the 999th row of Pascal's triangle and multiplying by the product of the terms in the 1001th row and dividing by the square of the product of the terms in the 1000th row!

c)   What transcendental number do you think this might be approaching?

# Triangles within Triangles

Modular arithmetic is sometimes called 'clock arithmetic'. What time will it be in 25hours from now? The answer is simply "one hour past the current time." The same is true for 49 hours from now and also for 73 hours. We ignore the number of days, the multiples of 24 and focus on the 'remainder'. The calculator is capable of performing modular arithmetic. For the examples given above:  Mod(25,24) = 1; Mod(49,24) = 1 and Mod(73,24) = 1. The first number in the Mod command is the quotient and the second number is the divisor, the answer is the 'remainder'.

**Question: 18**

Explore what happens if Mod(n,2) is applied to every element in Pascal's triangle?  You will need to create many lines to see the pattern.
**Hint:** Try colouring regions containing 1's in one colour and 0's in another.