To Infinity and Beyond

Math Objectives

- Develop an understanding of what it means to take a limit "at" infinity.
- Develop an understanding of behavior that prevents limits from occurring by means of chaos or oscillation.
- Estimate limits from graphs and tables of values.
- Connect the ideas of end behavior, horizontal asymptotes, and limits at infinity.
- Students will try to make a connection with how to understand these topics in IB Mathematics courses and on their final assessments.

Vocabulary

- Limits
- Infinity
- Asymptote

About the Lesson

- This lesson is aligning with the curriculum of IB Mathematics Applications and Interpretations SL/HL and IB Mathematics Approaches and Analysis SL/HL
- This falls under the IB Mathematics Content Topic 5 Calculus:
  5.1 (AI/AA SL/HL):
  (a) Introduction to the concept of a limit.
  5.12 (AA HL only):
  (b) Understanding of limits (convergence and divergence)

  As a result, students will:
  - Apply this information to real world situations.

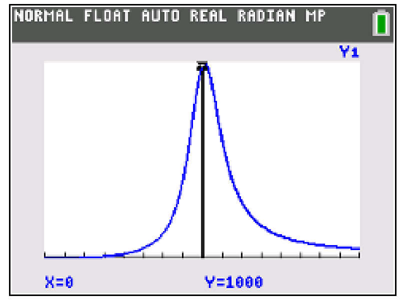
Teacher Preparation and Notes.

- This activity is done with the use of the TI-84 family as an aid to the problems.

Activity Materials

- Compatible TI Technologies: TI-84 Plus*, TI-84 Plus Silver Edition*, TI-84 Plus C Silver Edition, TI-84 Plus CE

** with the latest operating system (2.55MP) featuring MathPrint™ functionality.*



Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:
Student Activity
To Infinity and Beyond_Student-84CE.pdf
To Infinity and Beyond_Student-84CE.doc

The limit of a function is the behavior the output value approaches as the input value approaches a particular value. Limits are also examined when the output value keeps increasing or decreasing without bound. In this case the notation is $\lim_{x\to\infty}()$, as is read as “the limit as the input value approaches infinity.” The limit exists when the function moves toward a single output value and fails to exist when it does not move toward one value.

This activity has three parts. First, you will examine, graphically and numerically, the behavior of functions as the input approaches infinity. Next, you will examine graphically limits that do not exist because of continued chaotic output behavior as the input values continue to approach a particular value. Finally, you will examine a variety of limit problems.

Teacher Tip: Students should have an understanding of the imprecise nature of electronic utilities and what happens when the precision limits are reached. They should be able to manipulate graphs and tables of values manually and with the handheld.

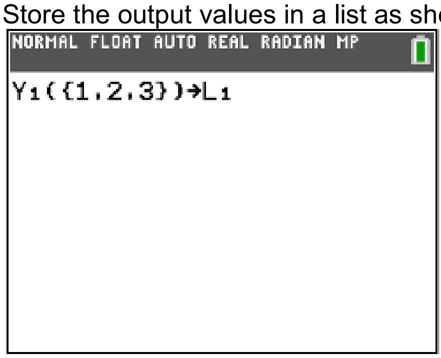
Teacher Tip: Students need to be careful as they often misinterpret infinity as an actual value to be substituted in a function. The may also incorrectly estimate infinity by pushing the graphing handheld beyond its precision limits and misinterpreting the result.

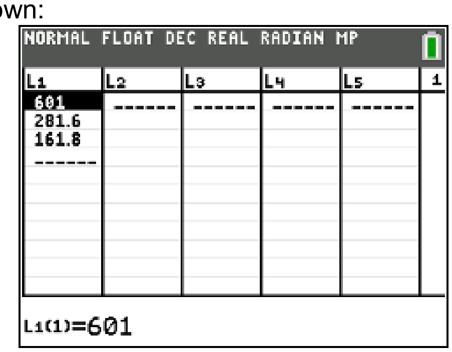
Problem 1

Input this function into **Y_1** in the **Y =** editor: $f(x) = \frac{2x^2+200x+1000}{x^2+1}$

(a) Record the function values for the inputs {1, 2, 3}. **Solution:** {601, 281.6, 161.8}

(b) Store the output values in a list as shown:





(c) Repeat the process for the input values {100, 200, 300}. Store the output values in **L_2**, and record the values.

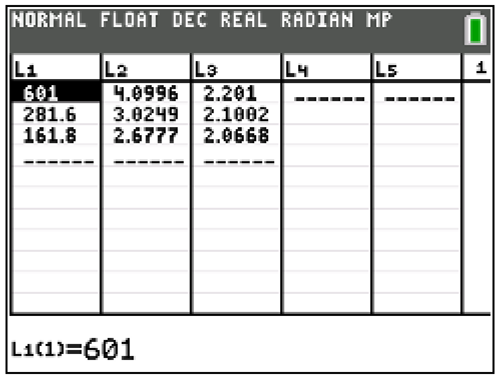
Solution: {4.0996, 3.0249, 2.6777}

(d) Repeat the process for the input values {1000, 2000, 3000}. Store the output values in **L_3**, and record the values.

Solution: {2.201, 2.1002, 2.0668}

(e) With a classmate, look at the values in all three sets and draw a conclusion regarding the behavior of the function.

Possible discussion: Answers will vary, but the data seems to be getting close to the value of 2.



(f) To confirm your conclusion, try the input values {10^{10}, 10^{15}, 10^{20}}, and record your results.

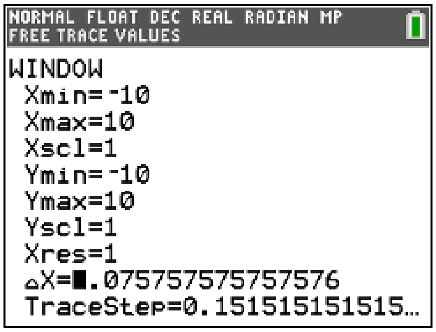
Solution: {2.00000002, 2, 2}

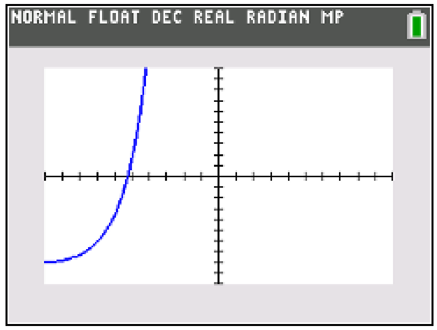
(g) With a classmate, discuss if the function actually reaches the exact value of the limit. Share with the class if you think this is a reasonable result.

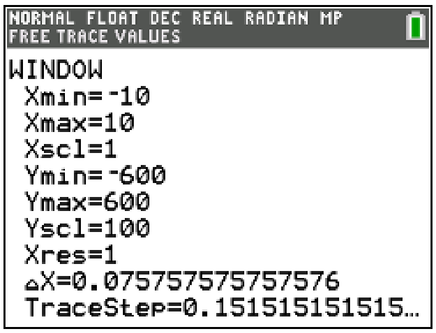
Possible discussion: Answers will vary. The question is leading students to say that it appears so; however, the function never actually reaches the value of 2.

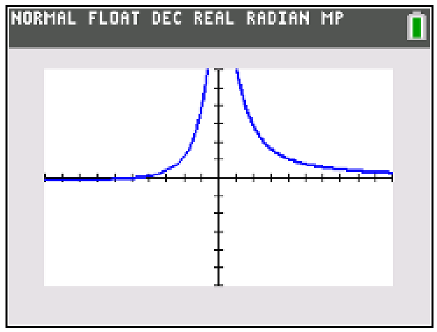
Teacher Tip: Care should be taken to distinguish between the value of $f(x)$ actually being 2 and the graphing handheld producing a value of 2 because of its precision capabilities. This is a good place for discussion of equals versus approaches very closely.

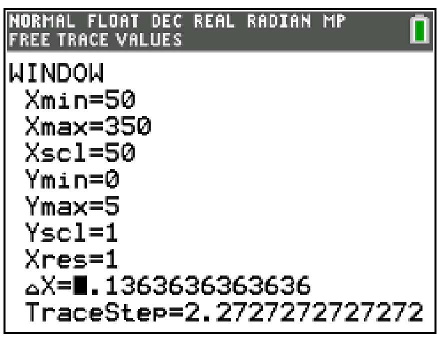
(h) Now let's look at the behavior graphically. Take several minutes exploring what viewing windows you would need to see the behavior from the input values listed in parts a, c, d, and f. Compare your windows on the handheld with a classmate and discuss what you notice.

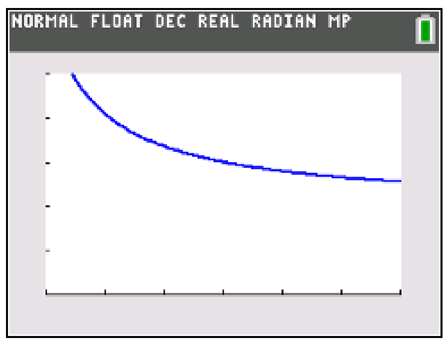


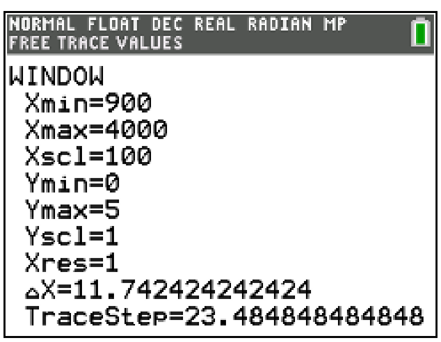


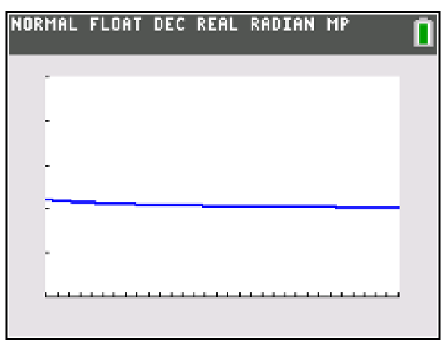












(i) Estimate: $\lim\limits_{x\to\infty} \dfrac{2x^2+200x+1000}{x^2+1}$ **Solution:** 2

(j) With a classmate, investigate and draw a conclusion of the following limit:

$$\lim_{x \to -\infty} \frac{2x^2+200x+1000}{x^2+1}$$

Possible discussion: The limit is 2, this function has a horizontal asymptote of y = 2 and the farther the curve approaches to the left and right, the closer it will get to the value of 2.

Teacher Tip: Teachers may want to make some connections here to previous course work (Precalculus and Algebra 2) with respect to horizontal asymptotes.

Problem 2

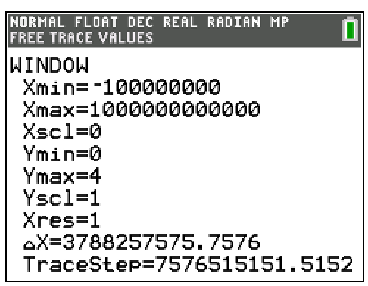
Now investigate the limit $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ as you did in **Problem 1**. Fill in the following table given the following input values:

X	L_1	X	L_2	X	L_3	X	L_4
1	**2**	100	**2.7048**	1000	**2.7169**	10^{10}	**2.71828**
2	**2.25**	200	**2.7115**	2000	**2.7176**	10^{15}	**1**
3	**2.3704**	300	**2.7138**	3000	**2.7178**	10^{20}	**1**

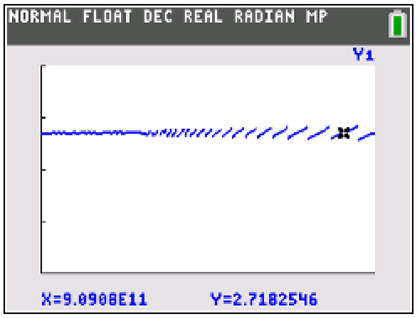
(a) With a classmate, discuss what you notice as the input values approach infinity.

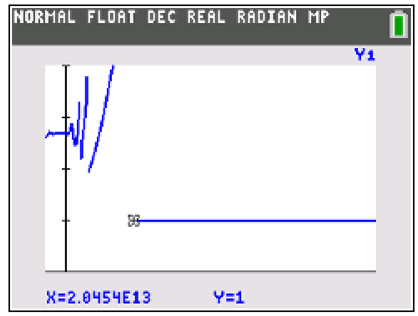
Possible discussion: Although the value of the limit is e, it is may or may not be expected that students will recognize such a value. A reasonable answer is 2.718.

(b) Given the following viewing window, graph the function and use the **trace** button to help describe what you see.



Possible discussion:





(c) State and explain the limit: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

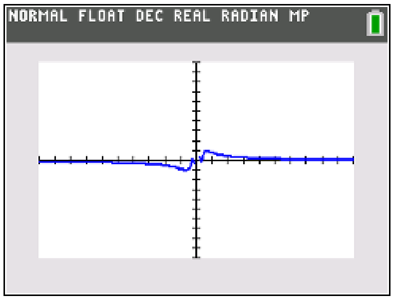
Possible discussion: Although the value of the limit is e, it is may or may not be expected that students will recognize such a value. A reasonable answer is 2.718, but at a certain point the curve begins to oscillate around the limit of 2.718, and if you increase the value of x toward infinity, at a certain point $\frac{1}{x} = 0$ and the limit will be 1 (see the graphs above).

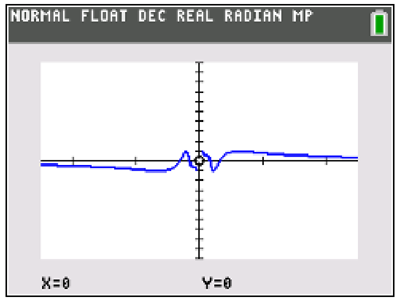
Problem 3

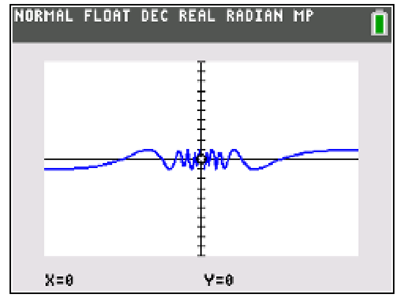
Before you practice limits on your own, let's examine the behavior of one more function: $f(x) = \sin\left(\frac{1}{x}\right)$ Use the standard window, make sure your handheld is in radian measure and graph $f(x)$. Please set your **Zoom Factors** by pressing **zoom**, **memory**, **4: SetFactors**. Make sure **XFact = 4** and **YFact = 1**.

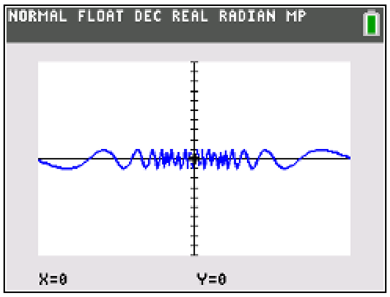
Discuss with a classmate what you notice about the function as the input values approach 0 and describe your results of the limit: $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$

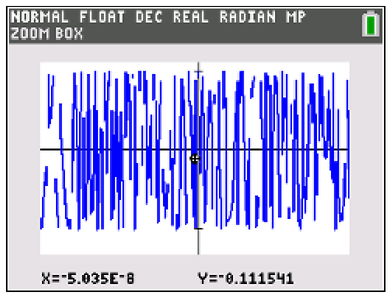
Possible discussion: The $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist. Although it does not move toward infinity or negative infinity, the function oscillates wildly at x = 0 and never gets near a single value (see the multiple zoomed in graphs below).











Practicing Limits

Evaluate the limit for each expression:

1. $\lim\limits_{x \to 0} \frac{\sin(x)}{x}$ **Solution:** 1

2. $\lim\limits_{x \to \infty} \frac{x-2}{2x+5}$ **Solution:** $\frac{1}{2}$

3. $\lim\limits_{x \to 0} \frac{|x|}{x}$ **Solution:** The limit does not exist. (There are two different limits as 0 is approached from different sides).

4. $\lim\limits_{x \to 0} \frac{\sin(5x)}{4x}$ **Solution:** $\frac{5}{4}$

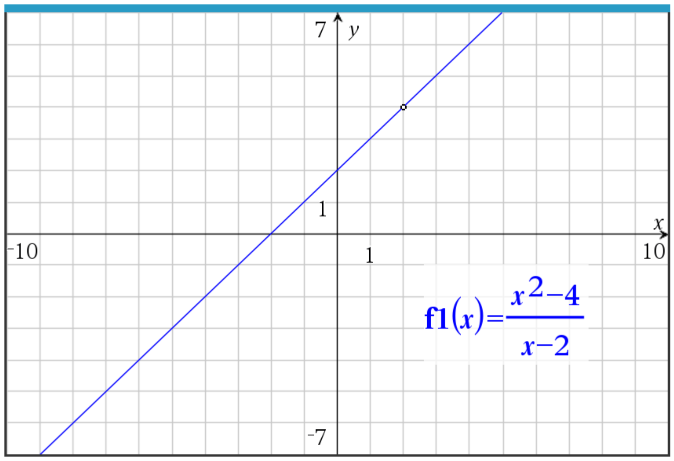
5. $\lim\limits_{x \to -\infty} \frac{4x-1}{\sqrt{x^2+2}}$ **Solution:** -4

6. $\lim\limits_{x \to 1} \frac{x^3-1}{x-1}$ **Solution:** 3

7. $\lim\limits_{x \to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4}\right)$ **Solution:** $\frac{1}{4}$

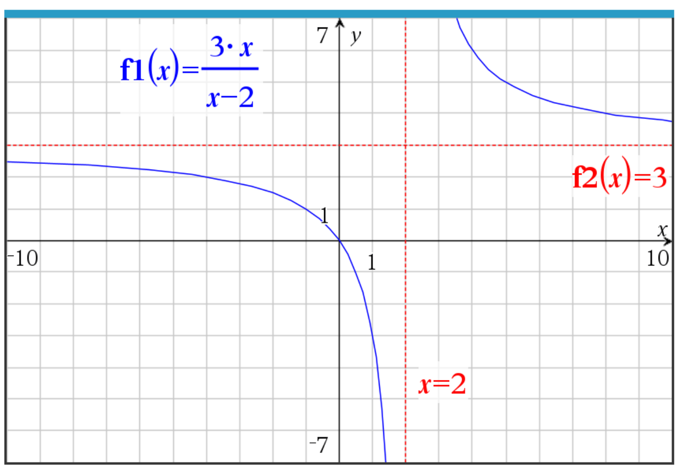
Further IB Applications

1. (a) Sketch the graph $y = \frac{x^2-4}{x-2}, x \neq 2$.



(b) Find $\lim\limits_{x\to 2} \frac{x^2-4}{x-2}$ numerically. **Solution:** 4

2. (a) Consider the function $f(x) = \frac{3x}{x-2}, x \neq 2$. Sketch the function.



(b) Evaluate the limit: $\lim\limits_{x\to\infty} \frac{3x}{x-2}$ **Solution:** 3

Teacher Tip: Please know that in this activity there is a lot of time dedicated to students talking with one another and sharing their thoughts with the class. The goal here is to not only review limits, but also to generate discussion.

***Note: This activity has been developed independently by Texas Instruments and aligned with the IB Mathematics curriculum, but is not endorsed by IB™. IB is a registered trademark owned by the International Baccalaureate Organization.*