Math Objectives

- Students will utilize the input and output of a function to determine its rule.
- Students will discover conditions that would restrict a function's domain.
- Students will understand that a composition utilizes the output of the inside function as the input of the outside function.
- Students will understand the special relationship between two inverse functions and their composition.
- Students will look closely to discern a pattern (CCSS Mathematics Practice).
- Students will use technology to visualize the results of varying assumptions, explore consequences, and compare predictions with data (CCSS Mathematics Practice).

Vocabulary

- composite functions
- composition
- domain
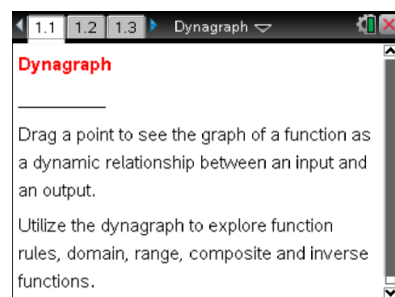- restricted domain
- inverse
- range

About the Lesson

- This lesson involves using a dynagraph to explore the relationship between the input and the output of a given function.
- As a result, students will:
- Discover how a composition utilizes the output of the inside function as the input of the outside function.
- Determine the rule for a function, as well as the domain and range of the function.
- Make and test predictions about the rule for two functions and for their composition.
- Discover the special relationship between two inverse functions and their composition.

TI-Nspire™ Navigator™

- Send the .tns file to students.
- Use Class Capture to formally assess students' understanding.
- Use Live Presenter for students to share their thinging.
- Use Quick Poll to assess students' understanding.

Activity Materials

- Compatible TI Technologies: TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software

Dynagraph

Drag a point to see the graph of a function as a dynamic relationship between an input and an output.

Utilize the dynagraph to explore function rules, domain, range, composite and inverse functions.

Tech Tips:

- This activity includes screen captures taken from the TI-Nspire CX handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire App. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:
Student Activity
- Dynagraph_Student.pdf
- Dynagraph_Student.doc

TI-Nspire document
- Dynagraph.tns

Discussion Points and Possible Answers

> **Tech Tip:** If students experience difficulty dragging a point, check to make sure that they have moved the cursor until it becomes a hand (✋) getting ready to grab the point. Then press `ctrl` 🖰 to grab the point and close the hand (✊). When finished moving the point, press `esc` to release the point.

> **Tech Tip:** If students experience difficulty dragging a point, tap on it once. This brings up a menu of options. Select the item you want to move and press Done. The point is now ready to move.

Move to page 1.3.

1. The dynagraph in the left work area represents an unknown function. As you drag the open point at the base of the arrow, notice that the *x* and *f(x)* values change, and the corresponding coordinate point, (*x, y*), is graphed on the Cartesian plane on the right.

   a. Use the table below to record the input and output values that you obtained.
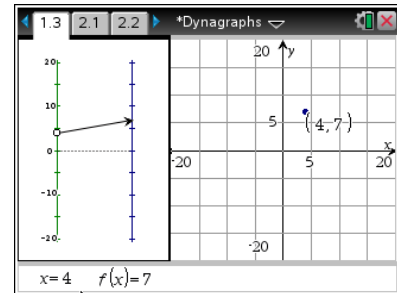
   Sample Answers: Student answers will vary, but all of the ordered pairs should satisfy the equation $f(x) = 2x - 1$.

x	−2	−1	0	1
y = f(x)	−5	−3	−1	1

   b. Find the "mystery rule" for *f(x)* that pairs each value for *x* with a value for *y*. Enter your equation into **f1** on the right side of the page. Test your prediction by using the dynagraph. Each point should lie on the graph of your function. Write the equation for the "mystery rule."

   Answer: The equation is $f1(x) = 2x - 1$.

> **Tech Tip:** Press `ctrl` `tab` to access the graph in the right work area. Press `ctrl` `G` to display the function entry line.

> **Tech Tip:** Double tapping within the right work area will display the function entry line. If students wish to edit a previous entry, they can double tap the right work area again and press the up arrow to access the **f1**(x) equation.

c.   What are the domain and the range of your function?

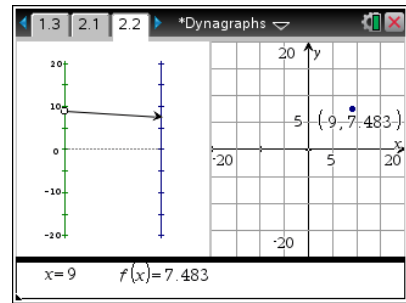**Answer:** Domain: all real numbers; Range: all real numbers.

TI-Nspire Navigator Opportunity: *Live Presenter and Quick Poll*
See Note 1 at the end of this lesson.

Move to page 2.2.

2.   The dynagraph on this page follows a non-linear function rule. As you drag the open point, you'll notice that the arrow disappears.

a.   Why does the arrow disappear? When does this happen?

Sample Answers: The arrow disappears when the function is undefined. This happens in the interval $-5 < x < 5$.

b.   Use the table below to record the input and output values that you obtained.

Sample Answers: Student answers will vary, but all of the ordered pairs should satisfy the equation $f(x) = \sqrt{x^2 - 25}$.

x	−13	−5	−2	0	2	5	8
$y = f(x)$	12	0	undefined	undefined	undefined	0	6.25

c.   Find the "mystery rule" for $f(x)$ that pairs each value for x with a value for y. Enter your equation in **f1**. After you've tested your prediction, write the equation for the "mystery rule."

Answer: The equation is $f1(x) = \sqrt{x^2 - 25}$.

d. What are the domain and the range of this function?

> **Answer:** Domain: $\{x \mid x \leq -5 \text{ or } x \geq 5\}$; Range: {Non-negative real numbers}

> **Teacher Tip:** Check that students are exploring the three intervals, $x \leq -5$, $-5 < x < 5$, and $x \geq 5$ to enable them to determine the function rule.


TI-Nspire Navigator Opportunity: *Class Capture and Quick Poll*
See Note 2 at the end of this lesson.

Move to page 3.2.

3. This dynagraph consists of three vertical number lines and behaves like two function machines wired in series. The point at x identifies a domain value on the first number line and is dynamically linked by the function $f(x)$ to a range value on the middle number line. That value is then linked by a second function $g(x)$ to a value $g(f(x))$ on the far right number line.
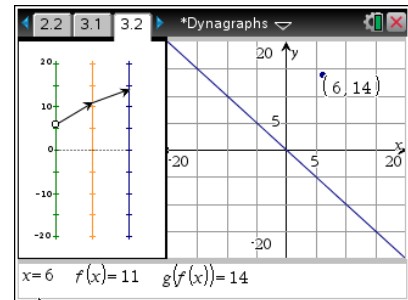
    For example, the number 5 is linked to 9 by $f(x)$ (because $f(5) = 9$), which in turn is linked to 12 by $g(x)$ (because $g(9) = 12$). Grab and drag the base of the arrow at point x. Set $x = 5$, and confirm that $f(5) = 9$ and $g(9) = 12$.

    a. Use the table below to record the input and output values that you obtained.

    > **Sample Answers:** Student answers will vary, but all of the ordered pairs for $f(x)$ should satisfy the equation **f1**$(x) = 2x - 1$. All of the ordered pairs for $g(f(x))$ should satisfy the equation **f2**$($**f1**$(x)) =$ **f1**$(x) + 3$.

x	5	3	1	0	−1	−3	−5
$f(x)$	9	5	1	−1	−3	−7	−11
$g(f(x))$	12	8	4	2	0	−4	−8

    b. Find the rule for **f1**$(x) = f(x)$ that pairs each value for x with a value for $f(x)$. Find the rule for **f2**$(x) = g(x)$ that pairs each value for x with a value for **f2**$($**f1**$(x)) = g(f(x))$. Enter your equations into **f1**(x) and **f2**(x) Test your predictions by using the dynagraph. Each point should lie on the graph of your function. Write the equations for the "mystery rules."

Answer: The equations are: **f1**(x) = 2x – 1 and **f2**(x) = x + 3.

📱💻**TI-Nspire Navigator Opportunity:** *Quick Poll*

See Note 3 at the end of this lesson.

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c. Find a rule for the single function **f3**(x) that gives the same result as **f2**(**f1**(x)) for all values of x. Enter your equation into **f3**(x). Test your prediction. Write the equation for this composite function.

  Answer: We obtain the equation **f3**(x) = 2x + 2 by substituting **f1**(x) = 2x - 1 into the equation for in **f2**(x). Thus, **f3**(x) = **f2**(**f1**(x)) = **f2**(2x – 1) = 2x – 1 + 3 = 2x + 2.

d. Compute and compare the following. You may want to use a calculator page to perform the computations.

  📟 **Tech Tip:** Calculate using the Scratchpad 🖩

  👆 **Tech Tip:** To add a calculator page, select ➕ > *Calculator*

  i. **f2**(**f1**(4))

  ii. **f1**(**f2**(4))

  Try other values of x. Does the order in which you apply the functions matter? Why do you think that this is true?

  Answers:
  i. **f2**(**f1**(4)) = 10

  ii. **f1**(**f2**(4)) = 13

  Yes, the order does matter. In **f2**(**f1**(4)), x = 4 is input into the function **f1**(x) = 2x – 1 to produce an output of 7. Then, 7 is input into the function **f2**(x) = x + 3 to produce an output of 10. In **f1**(**f2**(4)), x = 4 is input into the function **f2**(x) = x + 3 to produce an output of 7. Then, 7 is input into the function **f2**(x) = 2x – 1 produce an output of 13.

e. Give an example of two functions, $f(x)$ and $g(x)$, with neither function in the form $y = x$, such that $g(f(x)) = f(g(x))$.

Sample Answers: One such pair of functions is $f(x) = 3x - 1$ and $g(x) = \frac{1}{3}(x+1)$.

> **Teacher Tip:** The order of the functions does not matter if and only if $f(x)$ and $g(x)$ are inverses. Students will see this on Page 4.2.
>
> It might be difficult for students to come up with two functions that will answer this question correctly. However, if students have studied inverse functions in their previous courses, they might recall this on their own.
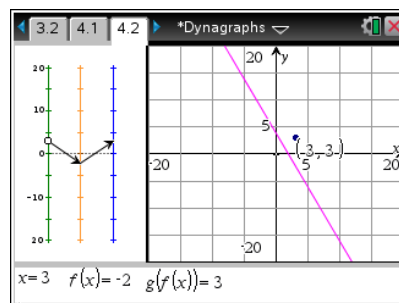

TI-Nspire Navigator Opportunity: *Class Capture*
See Note 4 at the end of this lesson.

Move to page 4.2.

4. The "inverse" of a function f, denoted f^{-1}, "undoes" the function—it maps a point y from the range back to its original x from the domain. You can think of a function and its inverse as a special case of function composition.

By definition, $g(x)$ is the inverse if and only if $g(f(x)) = x$ for every x in the domain of $f(x)$, and $f(g(x)) = x$ for every x in the domain of $f(x)$. In the context of the dynagraph, **f2** is the inverse of **f1** if **f2**(**f1**(x)) horizontally aligns with x for all values in the domain of **f1** (i.e. **f2**(**f1**(x)) = x), and vice versa.

a. The dynagraph on Page 4.2 shows the composite function $g(f(x))$. As you drag the open point, notice that $g(f(x)) = x$ for all input values of x.

Navigate to the right work area, and notice that **f1**(x) = $-2x + 4$ and **f2**(x) = x. Find the rule for **f2**, the inverse of **f1**. Enter your equation into **f2**(x). Test your prediction by using the dynagraph, and write the equation for **f2**.

Answer: We have **f2**(x) = **f1**$^{-1}$(x) = $\frac{-x}{2}$ + 2.

> **Tech Tip:** If either **f1**(x) or **f2**(x) is graphed in addition to the composite function, **f3**(x), hide the additional function(s). To hide a function, open the entry line and click on the up-chevron on the right. Select the function that you want to hide, and uncheck.

b. Evaluate **f2**(**f1**(*x*)) and **f1**(**f2**(*x*)).

   Answer: Since **f2**(*x*) = **f1**$^{-1}$(*x*), **f2**(**f1**(*x*)) = **f1**(**f2**(*x*) = *x*.


TI-Nspire Navigator Opportunity: *Quick Poll*

See Note 5 at the end of this lesson.

> **Teacher Tip:** The formal definition of an inverse function is given in this
> problem. You might want to provide students with several functions and
> have them identify the inverse. Encourage them to identify a function's
> inverse by switching the x and the y in the equation and solving for y. Be
> sure to reinforce that students must find both *f*(*g*(*x*)) and *g*(*f*(*x*)) to
> determine if two functions, *f* and *g*, are inverses. In problem 5, students will
> explore some of these subtleties.

Move to page 5.2.

5. a. Given the functions $f(x) = \sqrt{x}$ and $g(x) = x^2$, evaluate *f*(*g*(*x*))
       and *g*(*f*(*x*)). Are the functions inverses? Explain your answer.
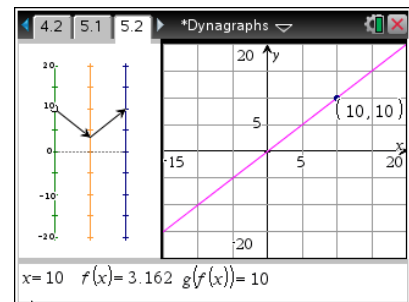
   Sample Answers: We see that, for $x \geq 0$, *g*(*f*(*x*)) = *f*(*g*(*x*)) = *x*. However, for *x* < 0, *f*(*x*) is
   undefined. Also, $f(g(x)) = f(x^2) = \sqrt{x^2}$. We know that $\sqrt{x^2} \neq x$, and that $\sqrt{x^2} = |x|$. Thus, *f*(*x*)
   and *g*(*x*) are not inverses.

b. On Page 5.2, grab and drag the base of the arrow at point *x*. What happens when *x* < 0? Why
   does this happen?

   Answer: When *x* < 0, the arrows disappear. This happens because *f*(*x*) is undefined when
   x < 0.

c. Navigate to the right work area, and notice that $\mathbf{f1}(x) = \sqrt{x}$ and $\mathbf{f2}(x) = x^2$. Find the rule for **f2**, the inverse of **f1**. Enter your equation into **f2**(x). Test your prediction by using the dynagraph, and write the equation for **f2**.

Answer: To obtain an inverse for **f1**(x), we have to restrict the domain of **f2**(x) to the set of non-negative real numbers. Thus, **f2**(x) = x^2 | $x \geq 0$.

> **Tech Tip:** To enter the equation with the restricted domain, type X x² ctrl = || X ctrl = ≥ 0 .

> **Tech Tip:** To enter the equation with the restricted domain, double tap the right work area to bring up the TI-Nspire keyboard and choose **ABC** to return to your device's native keyboard. The **|** (such that) symbol can be found in the punctuation (# + =) portion of your device's numeric keyboard. To place an inequality, return to the TI-Nspire keyboard and hold your finger over the **>** button to bring up more inequality options.

d. What are the domain and range of **f1**(x) and **f2**(x)?

Answer: f1(x): Domain and Range: {Non-Negative real numbers}
f2(x): Domain and Range: {Non-Negative real numbers}

Wrap Up

Upon completion of the discussion, the teacher should ensure that students are able to understand:

*   Conditions that restrict the domain and range of a function.

*   That a composition utilizes the output of the inside function as the input of the outside function.

*   That inverse functions are special cases of composition functions where the output of the composition function is the input of the original function.

*   That two functions, $f(x)$ and $g(x)$, are inverses if and only if $g(f(x)) = x$ for every x in the domain of $f(x)$, and $f(g(x)) = x$ for every x in the domain of $g(x)$.

Assessment

Have students find the outputs for several compositions of functions, as well as the rule for the composition. Also, ask students to determine the inverse of several functions.

TI-Nspire Navigator

Note 1

Question 1, Name of Feature: Live Presenter and Quick Poll Ask one of the students to serve as the *Live Presenter* and demonstrate to the class how to navigate through the activity. Use a *Quick Poll* to collect students' function rules, and assess their understanding.

Note 2

Question 2, Name of Feature: Class Capture and Quick Poll A *Class Capture* can be used to capture screen shots of the regions where the function is defined and undefined. An additional *Class Capture* or a *Quick Poll* can be used to compare students' function rules. You might want to leave *Class Capture* running in the background, with a 30 second automatic refresh, and without student names displayed. This will enable you to monitor students' progress and make the necessary adjustments to your lesson.

Note 3

Question 3b and 3c, Name of Feature: Quick Poll You might want to send a *Quick Poll* and have students send their equations for **f1** and **f2**. If students are having difficulty obtaining the equations, you can discuss how to obtain the equations. If most of the students have the correct equations for **f1** and **f2**, you might want to send another *Quick Poll* to have students send their equation for **f3**.

Note 4

Question 3e, Name of Feature: Class Capture This would be a good time to check *Class Capture* to assess students' ability to find two functions to answer this question. Be sure to give students enough time to make this discovery on their own. However, if they are really struggling and are getting frustrated, it might be a good idea to come back to this question after progressing to the next problem.

Note 5

Question 4, Name of Feature: Quick Poll You might want to send a *Quick Poll* to assess whether or not students were able to determine that $f2(f1(x)) = f1(f2(x) = x$.