## Math Objectives

- Students will understand that trigonometric functions of an angle do not depend on the size of the triangle within which the angle is contained, but rather on the ratios of the sides of the triangle.
- Students will understand the meaning of reference angles and use reference angles to determine the trigonometric functions of a given angle.
- Students will create an algorithm to determine the trigonometric functions of an angle based on its reference angle.
- Students will reason abstractly and quantitatively (CCSS Mathematical Practice).
- Students will look for and express regularity in repeated reasoning (CCSS Mathematical Practice).

## Vocabulary

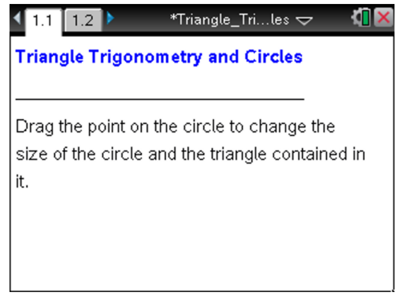
- hypotenuse
- horizontal and vertical legs
- trigonometric functions
- reference angle

## About the Lesson

- This lesson involves the ratios of side lengths of triangles with invariant angle measures and the relationships to trigonometric functions of an angle.
- As a result, students will:
  - Manipulate the radius of a circle to change the dimensions of a triangle with fixed angles and observe the resulting ratios of side lengths.
  - Make and test predictions about the impact of side lengths and the quadrant in which the reference triangle is located on the trigonometric functions of an angle.

## TI-Nspire™ Navigator™ System

- Transfer a File.
- Use Screen Capture to examine patterns that emerge.
- Use Quick Poll to assess students' understanding.




**TI-Nspire™ Technology Skills:**

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

**Tech Tips:**

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing ctrl G.

**Lesson Files:**

*Student Activity*

Triangle_Trigonometry_and_Circles_Student.pdf

Triangle_Trigonometry_and_Circles_Student.doc

*TI-Nspire document*

Triangle_Trigonometry_and_Circles.tns

Visit [www.mathnspired.com](http://www.mathnspired.com) for lesson updates and tech tip videos.

## Discussion Points and Possible Answers

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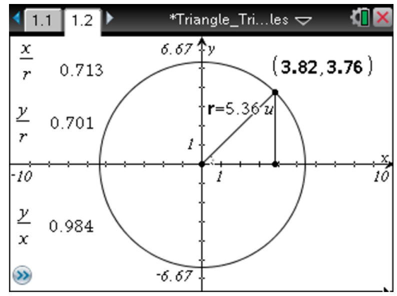



1. The angle in the right triangle which is adjacent to its horizontal leg (opposite its vertical leg) is fixed. You can drag the point on the circle to change the radius of the circle, and, therefore, the hypotenuse of the triangle. The length of the radius and the coordinates of the point are shown.
   a. Set the radius to 3. You can do this by clicking directly on the measurement for the radius and typing in 3. What are the lengths of all three sides of the triangle? How do you know?

   **Answer:** The horizontal leg is 2.14 units long, and the vertical leg is 2.1 units long. The hypotenuse is 3 units long. I know this because the coordinates of the point on the circle are (2.14, 2.1). The horizontal leg starts at the origin, so its length must be 2.14. The vertical leg starts at the x-axis, so its length must be 2.1. The hypotenuse of the triangle is the radius of the circle, so its length must be 3.

   b. Predict what will happen to the lengths of the horizontal and vertical legs of the triangle if you drag the point on the circle to change the radius. Why do you predict this?

   **Sample Answers:** The lengths of the horizontal and vertical legs will increase if r increases and decrease if r decreases. As r increases, the coordinates of the point increase, and the coordinates of the point correspond to the lengths of the other two legs of the triangle. Thus if r increases, the length of those legs will increase. The opposite is true if r decreases.

**TI-Nspire Navigator Opportunity: *Quick Poll***
**See Note 1 at the end of this lesson.**

   c. Test your prediction in part b. What happens to the side lengths of the triangle? Why?

   **Sample Answers:** As r increases, the side lengths increase. See part b for justification.

2. Set the radius to 3 again. The ratios of the side lengths appear on the left side of the screen.
   a. Predict what will happen to each ratio as r increases and decreases. Why do you predict this?

   **Sample Answers:** As r increases or decreases, the ratios will stay the same because the length of the horizontal and vertical legs will change in proportion to the length of the hypotenuse.

**TI-Nspire Navigator Opportunity: *Quick Poll***
**See Note 2 at the end of this lesson.**

   b. Test your prediction in part a. What happens to the ratios as r changes? Why?

   **Sample Answers:** The ratios stay the same. Because the angle opposite the vertical length is fixed, and the right angle is fixed, all triangles will be similar (by AA similarity). Thus the ratios of side lengths will always be the same.

3. Each of the ratios on the left side of the screen is the value of a trigonometric function for the angle θ—the interior angle of the triangle formed by the hypotenuse and the horizontal leg.
   a. Rewrite each ratio on the left side of the screen as a trigonometric function of θ. How did you determine these?

   **Answer:** $\frac{x}{r} = \cos\theta$, $\frac{y}{r} = \sin\theta$, and $\frac{y}{x} = \tan\theta$. In a triangle, $\cos\theta$ is the ratio of the side adjacent the angle to the hypotenuse, and x is the length of the horizontal side adjacent to the angle, while r is the length of the hypotenuse. The other two ratios are determined similarly.

   b. Based on your responses to question 2, what will happen to the trigonometric functions of θ as the side lengths of the triangle change, but the angle stays fixed? Why?

   **Sample Answers:** Each trigonometric function of θ will stay the same. This is because the function can be defined as the ratio of the triangle side lengths, and since the angle is fixed, the triangles will be similar; hence, side length ratios will be the same.

   c. Use a geometric argument to explain why tan θ will always be the same if θ is the same, no matter how large the sides of the triangle in which θ is contained.

   **Sample Answers:** If the angle θ remains fixed, any two triangles generated in this activity will be similar, as they will both share the common angle θ and the common right angle. Thus by AA similarity, any two right triangles sharing common angle θ will be similar, and so the ratios of their side lengths will be equal. Therefore, in any two right triangles with common angle θ, the ratio vertical leg : horizontal leg is equal. But this ratio is simply tan θ.

d. Does the same argument from part c hold for the other trigonometric functions? Explain.

**Sample Answers:** Yes, since the other trigonometric functions are equivalent to ratios of side lengths and any right triangles sharing common angle θ will be similar. Thus the ratios of their side lengths, the values of the trigonometric functions, will be equal.

4. a. What happens to the triangle if you drag the point around the circle? What other triangles are possible?

**Answers:** There are three other types of triangles possible; one lying in the second quadrant, one in the third, and one in the fourth, up to differences in side lengths.

**TI-Nspire Navigator Opportunity: *Screen Capture***
**See Note 3 at the end of this lesson.**

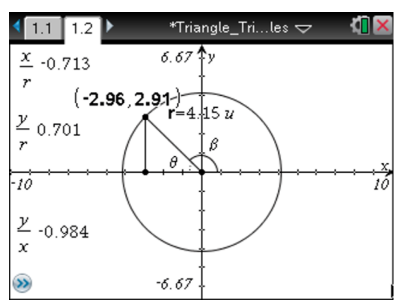
b. Why are these the possible triangles?

**Sample Answers:** If the angle remains fixed, the only possible differences would be orientations. Maintaining the measure of θ will only allow for either changes in the length of r or reflections of the original triangle about one or both of the axes.

c. What do all these triangles have in common?

**Sample Answers:** They are all similar; the angles are all the same measure.

5. An angle formed by the x-axis and a segment or ray in the coordinate plane is measured counter-clockwise from the x-axis. Move the point on the circle so the triangle appears in the second quadrant. Consider the angle from the x-axis counter-clockwise to the hypotenuse of the triangle, marked β in the figure to the right.



a. The acute angle between the x-axis and the ray or segment in the coordinate plane is called the reference angle. θ is the reference angle for β. What is the relationship between θ and β? Express β in terms of θ if both are measured in radians. Express β in terms of θ if both are measured in degrees. Fill in the table below.

**Answer:** The relationships are summarized in the table below.

| | β in quadrant I | β in quadrant II | β in quadrant III | β in quadrant IV |
|---|---|---|---|---|
| Degrees | $\beta = \theta$ | $\beta = 180 - \theta$ | $\beta = 180 + \theta$<br>$\beta = \theta - 180$ | $\beta = 360 - \theta$<br>$\beta = -\theta$ |
| Radians | $\beta = \theta$ | $\beta = \pi - \theta$ | $\beta = \pi + \theta$<br>$\beta = \theta - \pi$ | $\beta = 2\pi - \theta$<br>$\beta = -\theta$ |

b. How can you use the ratios displayed to find tan β, sin β, and cos β? Explain.

**Answer:** The ratios displayed are equivalent to tan β, sin β, and cos β, respectively up to sign. The sign can be determined based on the quadrant in which β lies. If β is in quadrant I, all trigonometric functions are given by their respective ratios. If β is in quadrant II, the tangent and cosine are negative, but of the same magnitude as the ratios. If β is in quadrant III, the cosine and sine are negative, but of the same magnitude as the ratios. If β is in quadrant IV, the sine and tangent are negative, but of the same magnitude as the corresponding ratios.

6. a. What will happen to the three ratios if the triangle is in the third quadrant? The fourth quadrant? Explain.

**Sample Answers:** The ratios might change signs. In the third quadrant, $x/r$ and $y/x$ will be negative. In the fourth quadrant, $x/r$ and $y/r$ will be negative.

b. Move the point on the circle to test your predictions in part a. How do the results compare to your predictions?

**Sample Answers:** Results confirm the predictions.

7. Write an algorithm (a set of steps) explaining how to find sin β, cos β, and tan β based on the reference angle θ. Your algorithm should address all possible locations of β and the differences (if any) when β is measured in radians or degrees.

**Sample Answers:**

Step one: Determine the quadrant in which β lies.

Step two: Determine the reference angle, θ, for β.

Step two a: If β is in quadrant I, it is its own reference angle.

Step two b: If β is in quadrant II, the reference angle is 180 – β, or π – 180.

Step two c: If β is in quadrant III, the reference angle is β – 180, or β – π.

Step two d: If β is in quadrant IV, the reference angle is 360 – β, or 2 π – β.

Step three: Determine the value of the desired trigonometric function of the reference angle θ.

Step four: Determine the sign of the trigonometric function of β.

Step four a: If β is in quadrant I, sin β = sin θ, cos β = cos θ, and tan β = tan θ

Step four b: If β is in quadrant II, sin β = sin θ, cos β = -cos θ, and tan β = -tan θ

Step four c: If β is in quadrant III, sin β = - sin θ, cos β = -cos θ, and tan β = tan θ

Step four d: If β is in quadrant IV, sin β = - sin θ, cos β = cos θ, and tan β = -tan θ

## Wrap Up

Upon completion of the discussion, the teacher should ensure that students are able to understand:

- That the values of trigonometric functions depend on the angle and the ratios of the side lengths rather than the individual side lengths.
- That the trigonometric functions of an interior acute angle of a right triangle are ratios of side lengths of that triangle.
- That the values of trigonometric functions of angles can be found using the values of trigonometric functions of acute reference angles.

## Assessment

Provide students with several angles larger than 90° or of negative measure and have them use reference angles to determine the values of trigonometric functions evaluated at those angles.

## TI-Nspire Navigator

**Note 1**

**Question 1, Name of Feature: Quick Poll**

Ask students to predict: if r increases, will x increase, decrease, or stay the same? What about y? This can generate a discussion about student predictions and their justifications for their predictions and provide the teacher with an indication of what, if any, misconceptions to check for after students explore.

**Note 2**

**Question 2, Name of Feature: Quick Poll**

Ask students to predict: If r increases, will x/r increase, decrease, or stay the same? What about the other two ratios? This can generate a discussion about student predictions and their justifications for their predictions and provide the teacher with an indication of what, if any, misconceptions to check for after students explore.

**Note 3**

**Question 4, Name of Feature: Screen Capture**

A Screen Capture can be used to display the different possible orientations of triangles in the coordinate plane with the same angle measures, and be used to demonstrate the location of the reference angles.