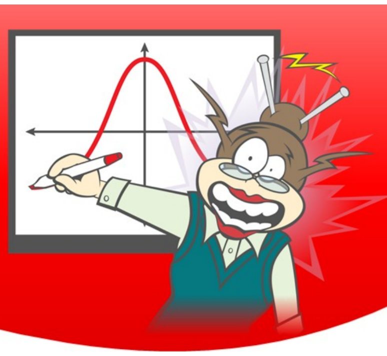
STUDENT REVISION SERIES



Applications of Differential Equations: Inflow – Outflow Mixing Problems

Question: 1.

A tank initially holds 30 L of water in which 3 kg of salt has been dissolved. Pure water is poured into the tank at a rate of 8 L per minute. The mixture in the tank is stirred continuously and flows out of the tank at a rate of 4 L per minute.

a. Show that the differential equation for Q, the number of kilograms of salt in the tank after t minutes, is given by $\frac{dQ}{dt} = \frac{-4Q}{30+4t}$

b. Find Q in terms of t.

c. Find the amount of salt that has flowed out of the tank over the first 10 minutes. Give your answer correct to 1 decimal place.

Question: 2.

A tank initially contains 50 litres of water with 5kg of salt dissolved in it. A salt solution with a concentration of 0.2 kg/L flows into the tank at a rate of 3L/min. The mixture is stirred uniformly and flows out at a rate of 3L/min. Let x be the amount of salt in the tank after t minutes.

a. Show that the differential equation that describes this scenario is given by

$$\frac{dx}{dt} = \frac{30 - 3x}{50}$$

b. Solve this differential equation to find x in terms of t

c. Calculate the amount if salt in the tank after 3 minutes. Give your answer correct to 2 decimal places.

d. Sketch the graph of x against t.

Question: 3.

An irrigation tank contains 2000L of water that initially has 100kg of a soluble fertilizer dissolved in it. The well-mixed solution is pumped out at a rate of 5 L/min. How much fertilizer is in the tank 20 min after the pumping process begins?

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Question: 4.

A water storage plant has discovered that one of their 9000L water tanks has a chlorine concentration of 5 mg/L, above the safe limit of 3.5 mg/L. To reduce the chlorine concentration, pure water is pumped into the tank at a rate of 30 L/min and is pumped out at a rate of 60 L/min. Let x be the total amount of chlorine (mg) in the tank at any time t (min).

a. Show that $\frac{dx}{dt} = \frac{-2x}{300-t}$

b. Find x in terms of t

c. i. Find the volume of water in the tank when the concentration reaches 4 mg/L

c. ii. When the concentration in the tank reaches 4 mg/L, the water being pumped into the tank is switched to chlorine -treated water with a concentration of 3.5 mg/L, and the rate of inflow is increased to 60 L/min. Show that the new differential equation describing this scenario is given by $\frac{dx}{dt} = 180 - \frac{4x}{585}$

c. iii. Find how long it takes, following the switch to chlorine -treated water being pumped in, for the chlorine concentration in the tank to first drop below the safe limit. Give your answer in the form $a\log_e(b)$, where a and b are positive integers.

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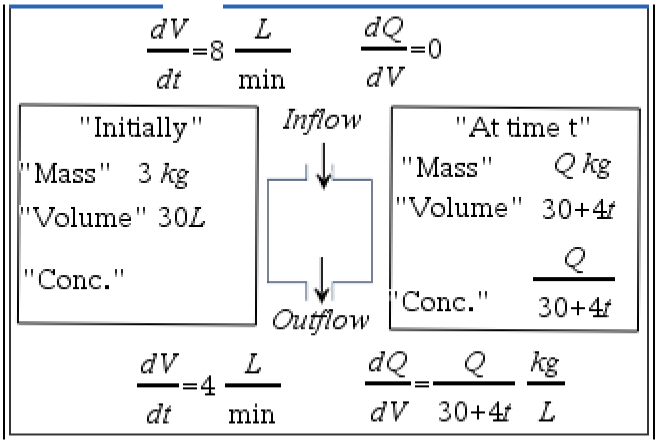
Answers

Question 1

a.



$$\frac{dQ}{dt} = \frac{dQ}{dt}_{in} - \frac{dQ}{dt}_{out}$$

$$\frac{dQ}{dt} = (8 \times 0) - 4 \times \frac{Q}{30 + 4t}$$

$$\frac{dQ}{dt} = -\frac{4Q}{30 + 4t}$$

b. $Q(t) = \frac{45}{2t+15}$, where the initial condition is $Q(0) = 3$.

$$\text{deSolve}\left(q' = -4\cdot\frac{q}{30+4\cdot t} \text{ and } q(0)=3, t, q\right)$$

$$q = \frac{45}{2\cdot t+15}$$

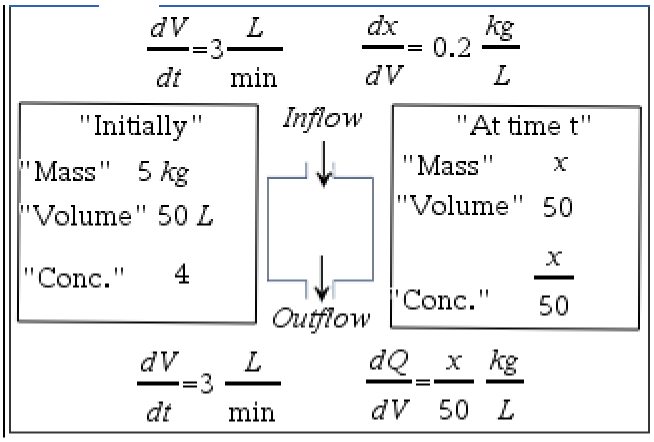
c. We wish to measure the amount of salt that has **flowed out** of the tank, which is given by $\frac{dQ}{dt}_{out}$

$$\int_0^{10} \left(4\cdot\frac{q}{30+4\cdot t}\right) dt \,\Big|\, q = \frac{45}{2\cdot t+15} \qquad 1.71429$$

$$\int_0^{10} \left|\frac{dQ}{dt}_{out}\right| dt = 1.7\ kg$$

Question 2

a.



$$\frac{dx}{dt} = \frac{dx}{dt}_{in} - \frac{dx}{dt}_{out}$$

$$\frac{dx}{dt} = (3 \times 0.2) - 3 \times \frac{x}{50}$$

$$\frac{dx}{dt} = \frac{30 - 3x}{50}$$

b.

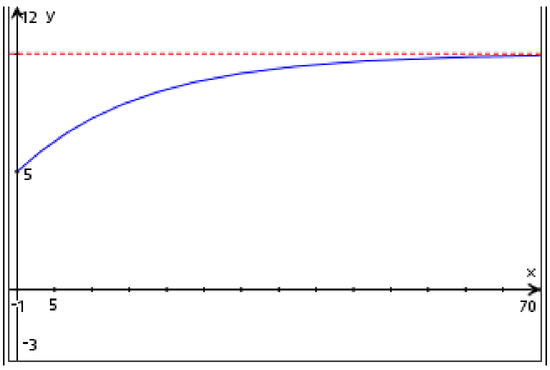
$$\text{deSolve}\left(x' = \frac{30-3\cdot x}{50} \text{ and } x(0)=5, t, x\right)$$

$$x = 10 - 5\cdot e^{\frac{-3\cdot t}{50}}$$

c. $x = 5.82$

$$x = 10 - 5\cdot e^{\frac{-3\cdot t}{50}} \,|\, t=3 \qquad x = 5.82365$$

d. .



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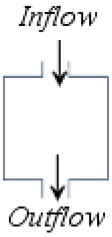
Question 3

$$\frac{dV}{dt}=0\frac{L}{\min} \qquad \frac{dx}{dV}=0\ \frac{kg}{L}$$

"Initially"
"Mass" 100 kg
"Volume" 2000 L
"Conc."



"At time t"
"Mass" x
"Volume" $2000-5t$
"Conc." $\frac{x}{2000-5t}$

$$\frac{dV}{dt}=45\frac{L}{\min} \qquad \frac{dQ}{dV}=\frac{x}{2000-5t}\ \frac{kg}{L}$$

$$\frac{dx}{dt}=\frac{dx}{dt}_{in}-\frac{dx}{dt}_{out}$$

$$\frac{dx}{dt}=(0\times 0)-5\times\frac{x}{2000-5t}$$

$$\frac{dx}{dt}=-\frac{5x}{2000-5t}$$

$$\text{deSolve}\left(x'=\frac{-5\cdot x}{2000-5\cdot t} \text{ and } x(0)=100,t,x\right) \qquad x=\frac{-(t-400)}{4}$$

$$x=\frac{-(t-400)}{4}|t=20 \qquad x=95$$

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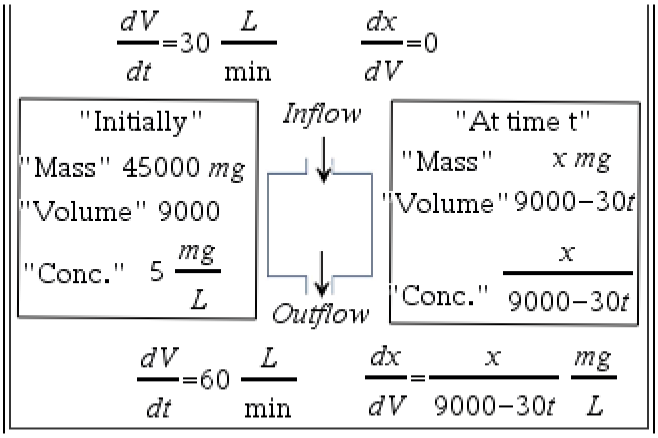
Question 4

a.



$$\frac{dx}{dt} = \frac{dx}{dt_{in}} - \frac{dx}{dt_{out}}$$

$$\frac{dx}{dt} = (30 \times 0) - 60 \times \frac{x}{9000 - 30t}$$

$$\frac{dx}{dt} = -\frac{2x}{300 - t}$$

b.

$$\text{deSolve}\left(x' = -2 \cdot \frac{x}{300-t} \text{ and } x(0) = 45000, t, x\right)$$

$$x = \frac{(t-300)^2}{2}$$

c. i. Volume is 7200 L

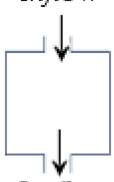
$$\text{solve}\left(\frac{x}{9000-30 \cdot t} = 4, t\right) \Big| x = \frac{(t-300)^2}{2} \qquad t=60$$

$$v = 9000 - 30 \cdot t \,|\, t = 60 \qquad v = 7200$$

c. ii.

$$\frac{dV}{dt}=60\frac{L}{\text{min}} \qquad \frac{dx}{dV}=3\ \frac{mg}{L}$$

"Initially"	
"Mass"	28800
"Volume"	7200
"Conc."	4



"At time t"	
"Mass"	x
"Volume"	7200
"Conc."	$\frac{x}{7200}$

$$\frac{dV}{dt}=60\frac{L}{\text{min}} \qquad \frac{dQ}{dV}=\frac{x}{7200}\frac{mg}{L}$$

$$\frac{dx}{dt}=\frac{dx}{dt}_{in}-\frac{dx}{dt}_{out}$$

$$\frac{dx}{dt}=(60\times 3)-60\times\frac{x}{7200}$$

$$\frac{dx}{dt}=180-\frac{x}{120}$$

c. iii.

$$\text{deSolve}\left(x'=180-\frac{x}{120} \text{ and } x(0)=28800,t,x\right)$$

$$x=7200\cdot e^{\frac{-t}{120}}+21600$$

$$\text{solve}\left(\frac{x}{7200}=3.5,x\right) \qquad x=25200.$$

We need to know the amount of chlorine in the tank that would correspond to a concentration of 3.5 ml/L

$$\text{solve}\left(25200=7200\cdot e^{\frac{-t}{120}}+21600,t\right)$$

$$t=120\cdot\ln(2)$$

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